Numerical simulation of shallow water dam break flow problem

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Abstract—In this study we perform numerical simulation of shallow water equations for dam break flow problem. Lax Friedrichs and Lax Wenroff numerical methods are applied for the numerical solution of the shallow water equations. We estimate water height and water velocity for the test case of shallow water-dam break flow problem. The comparisons between the analytical and numerical results for 1-D dam break problem have been shown.

Keywords: Lax Wendroff, Shallow water dam break flow, Water height and water velocity.

1 Introduction

Shallow water dam break flow models are used in all kinds of applications such as flood warning system, impact of changes of water system and climate predictions. The analysis of dam break flow is important to capture spatial and temporal evolution of flood event and safety analysis. The dam break basically is catastrophic failure of dam, leading to uncontrolled release of water causing flood in the downstream region. Many Numerical methods are available in the literature to solve the SWE. Bellos et al. [1] reported a two dimensional numerical methods for dam break problem by using the combination of finite element and finite difference method. Fayssal and Mohammed [2] proposed a new simple finite volume method for the numerical solution of shallow water equations. Bagheri and Das [4] developed an implicit high order compact scheme for shallow water equations with dam break problem. Saiduzzaman & Ray [6] examined some numerical methods for shallow water equations. Ahmed et al. [7] developed Godunov type finite volume method for dam break problem.

In section 2, present a Shallow water dam break flow problem. We apply the Lax Friedrichs and Lax Wendroff methods to obtain the numerical solution for the Shallow water equations in section 3. In section 4, we present an algorithm for the numerical solution and develop a computer programming code for the implementation of the numerical scheme. Numerical results are present in section 5. Finally the conclusions of the work are given in the last section.

2 Governing equation

The Mathematical model that describes the water flow in a river are defined as

$$\frac{\partial h}{\partial t}+\frac{\partial}{\partial x}(hv)=0$$
$$\frac{\partial v}{\partial t}+\frac{\partial}{\partial x}(\frac{1}{2}v^2+gh)=0 \qquad (1)$$

Where $h(x,t)$ is the water height at the time t and at the space g is the acceleration due to gravity, $v(x,t)$ is the flow velocity in the x -direction.

We set the following initial condition and boundary condition
The initial condition:

$$h(x,0)=\begin{cases} h_l & ,x<=L/2 \\ h_r & ,others \end{cases} \qquad (2)$$
$$v(x,0)=v_0(x);\ 0\le x\le L$$

Neumann Boundary condition:

$$h_x(x_a,t)=h_a(t);\ h_x(x_b,t)=h_b(t);$$
$$v_x(x_a,t)=v_a(t);\ v_x(x_b,t)=v_b(t);$$

Where

h_l, h_r, $v_0(x)$, $h_a(t)$, $h_b(t)$, $v_a(t)$, $v_b(t)$ are known constant values.

3 Numerical methods

We present here the discretization of the shallow water dam break flow model by finite difference formula, which leads to formulate explicit finite difference methods for the numerical solution of the governing equation as a nonlinear partial differential equation.

3.1 Lax friedrichs method for shallow water equation

Consider the SWE equations (1) in the following vector form

$$\frac{\partial u}{\partial t}+\frac{\partial f(u)}{\partial x}=0 \qquad (3)$$

where

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$$u = \begin{pmatrix} h \\ v \end{pmatrix}$$

and

$$f(u) = \begin{pmatrix} hv \\ \frac{1}{2}v^2 + gh \end{pmatrix}$$

The numerical discretization of (3) is as follows

$$u_i^{n+1} = \frac{1}{2}(u_{i+1}^n + u_{i-1}^n) - \frac{\Delta t}{2\Delta x}(f_{i+1}^n - f_{i-1}^n)$$

Implies that

$$h_i^{n+1} = \frac{1}{2}(h_{i+1}^n + h_{i-1}^n) - \frac{1}{2}\frac{\Delta t}{\Delta x}(h_{i+1}^n v_{i+1}^n - h_{i-1}^n v_{i-1}^n)$$

$$v_i^{n+1} = \frac{1}{2}(v_{i+1}^n + v_{i-1}^n) - \frac{1}{2}\frac{\Delta t}{\Delta x}[(\frac{1}{2}v_{i+1}^n v_{i+1}^n + gh_{i+1}^n) - (\frac{1}{2}v_{i-1}^n v_{i-1}^n + gh_{i-1}^n)]$$

3.2 Lax Wendroff method

To begin with derivation of Lax Wendroff method, Consider the following Taylors series for $u(t+\Delta t, x)$.

$$u(t+\Delta t, x) = u(t,x) + \Delta t\frac{\partial u}{\partial t} + \frac{\Delta t^2}{2}\frac{\partial^2 u}{\partial t^2} + O(\Delta t^3)$$

(4)

Where the time derivatives can be replaced by space derivatives using $u_t + (f(u))_x = 0$. This has been done by so called Cauchy Kowalewski technique, which implies $\frac{\partial u}{\partial t} = -\frac{\partial f(u)}{\partial x}$

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial t}(-\frac{\partial f(u)}{\partial x}) = \frac{\partial}{\partial x}(-\frac{\partial f(u)}{\partial t})$$

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial x}(-f'(u)\frac{\partial u}{\partial t})$$

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial x}[f'(u)\frac{\partial f(u)}{\partial x}]$$

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial x}(A(u)\frac{\partial f(u)}{\partial x})$$

Where

$$A(u) \equiv \frac{\partial f(u)}{\partial u}$$

Substitute the preceding expressions of time derivatives (4) into the Taylors series of $u(t+\Delta t, x)$ to obtain

$$u(t+\Delta t, x) = u(t,x) - \Delta t\frac{\partial f(u)}{\partial x} + \frac{\Delta t^2}{2}\frac{\partial}{\partial x}(A(u)\frac{\partial f(u)}{\partial x}) + O(\Delta t^3) \qquad (5)$$

$$\frac{\partial f(u)}{\partial x} = \frac{f(u_{i+1}^n) - f(u_{i-1}^n)}{2\Delta x} + O(\Delta x^2)$$

$$\frac{\partial}{\partial x}(A(u)\frac{\partial f(u)}{\partial x})(t^n, x_i) = \frac{A(u_{i+\frac{1}{2}}^n)(f(u_{i+1}^n) - f(u_i^n)) - A(u_{i-\frac{1}{2}}^n)(f(u_i^n) - f(u_{i-1}^n))}{\Delta x^2} + O(\Delta x^2)$$

The resulting method is as follows

$$u_i^{n+1} = u_i^n - \frac{\Delta t}{2\Delta x}(f(u_{i+1}^n) - f(u_{i-1}^n)) + \frac{\Delta t^2}{2\Delta x^2}[A_{i+\frac{1}{2}}(f(u_{i+1}^n) - f(u_i^n)) - A_{i-\frac{1}{2}}(f(u_i^n) - f(u_{i-1}^n))]$$

Where

$$A_{i\pm\frac{1}{2}} = \frac{1}{2}(u_{i\pm1}^n + u_i^n)$$

In shallow water equation we have

$$u = \begin{pmatrix} h \\ v \end{pmatrix}, \quad f(u) = \begin{pmatrix} hv \\ \frac{1}{2}v^2 + gh \end{pmatrix}, \quad A(u) = f'(u) = \begin{pmatrix} v & h \\ g & v \end{pmatrix}.$$

$$\begin{pmatrix} h \\ v \end{pmatrix}_i^{n+1} = \begin{pmatrix} h \\ v \end{pmatrix}_i^n - \frac{\Delta t}{2\Delta x}\begin{pmatrix} f_{1i+1}^n - f_{1i-1}^n \\ f_{2i+1}^n - f_{2i-1}^n \end{pmatrix} + \frac{\Delta t^2}{2\Delta x^2}\left[\begin{pmatrix} v & h \\ g & v \end{pmatrix}_{i+\frac{1}{2}}\begin{pmatrix} f_{1i+1}^n - f_{1i}^n \\ f_{2i+1}^n - f_{2i}^n \end{pmatrix} - \begin{pmatrix} v & h \\ g & v \end{pmatrix}_{i-\frac{1}{2}}\begin{pmatrix} f_{1i}^n - f_{1i-1}^n \\ f_{2i}^n - f_{2i-1}^n \end{pmatrix}\right]$$

$$\begin{pmatrix} h \\ v \end{pmatrix}_i^{n+1} = \begin{pmatrix} h \\ v \end{pmatrix}_i^n - \frac{\Delta t}{2\Delta x}\begin{pmatrix} f_{1i+1}^n - f_{1i-1}^n \\ f_{2i+1}^n - f_{2i-1}^n \end{pmatrix} + \frac{\Delta t^2}{2\Delta x^2}\left[\begin{pmatrix} v_{i+\frac{1}{2}}(f_{1i+1}^n - f_{1i}^n) + h_{i+\frac{1}{2}}(f_{2i+1}^n - f_{2i}^n) \\ g(f_{1i+1}^n - f_{1i}^n) + v_{i+\frac{1}{2}}(f_{2i+1}^n - f_{2i}^n) \end{pmatrix} - \begin{pmatrix} v_{i-\frac{1}{2}}(f_{1i}^n - f_{1i-1}^n) + h_{i-\frac{1}{2}}(f_{2i}^n - f_{2i-1}^n) \\ g(f_{1i}^n - f_{1i-1}^n) + v_{i-\frac{1}{2}}(f_{2i}^n - f_{2i-1}^n) \end{pmatrix}\right]$$

$$h_i^{n+1} = h_i^n - \frac{\Delta t}{2\Delta x}(f_{1i+1}^n - f_{1i-1}^n) + \frac{\Delta t^2}{2\Delta x^2}[v_{i+\frac{1}{2}}(f_{1i+1}^n - f_{1i}^n) - v_{i-\frac{1}{2}}(f_{1i}^n - f_{1i-1}^n) + h_{i+\frac{1}{2}}(f_{2i+1}^n - f_{2i}^n) - h_{i-\frac{1}{2}}(f_{2i}^n - f_{2i-1}^n)]$$

$$v_i^{n+1} = v_i^n - \frac{\Delta t}{2\Delta x}(f_{2i+1}^n - f_{2i-1}^n) + \frac{\Delta t^2}{2\Delta x^2}[v_{i+\frac{1}{2}}(f_{2i+1}^n - f_{2i}^n) - v_{i-\frac{1}{2}}(f_{2i}^n - f_{2i-1}^n) + g(f_{1i+1}^n - 2f_{1i}^n + f_{1i-1}^n)]$$

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4 Algorithm for the Numerical Solution

To find out the numerical solution of the Governing model (1), we have to accumulate some variables which are offered in the following algorithm.

Input: nx and nt the number of spatial and temporal mesh points respectively.

t_f , the right end point of $(0,T)$

x_d , the right end point of $(0,b)$

v_0 , the initial velocity, apply as a initial condition

v_a , Left hand boundary condition

v_b , Right hand boundary condition

h_0 , the initial height, apply as a initial condition

h_a , Left hand boundary condition

h_b , Right hand boundary condition

Output: $v(x,t)$ the solution matrix

$h(x,t)$ the solution matrix

Initialization: $dt = \frac{T-0}{nt}$, the temporal grid size $dx = \frac{b-0}{nx}$ the spatial grid size

Step1. Calculation for numerical solution of SWEs by finite difference scheme

For $n=1$ to nt

For $i=2$ to nx

$u(n+1,i) = (1/2)*(u(n,i-1)+u(n,i+1)) - (dt/(2*dx))(f(n,i+1) - f(n,i-1))$

end

end

Step2: Output: $v(x,t)$, $h(x,t)$

Step3: Figure Presentation

Step4: Stop

5 RESULTS AND DISCSSIONS

We implement the numerical methods to simulate water depth and water velocity at different times and different points through shallow water dam break problem. We start by assuming that in both sides of the dam there are water with corresponding heights $h_l = 10m$ and $h_r = 5m$ (height ratio $h_r / h_l = 0.5$ in (2). A channel with 2000m length is considered. The dam is situated at 1000m downstream in channel. The initial velocity $v = 0m/s$ and the simulation time is 100s. Comparisons are carried out only for wet bed condition with respect to velocity and water height. Numerical simulations are performed by considering size $\Delta t = 0.5$. The results are shown in the following figures.

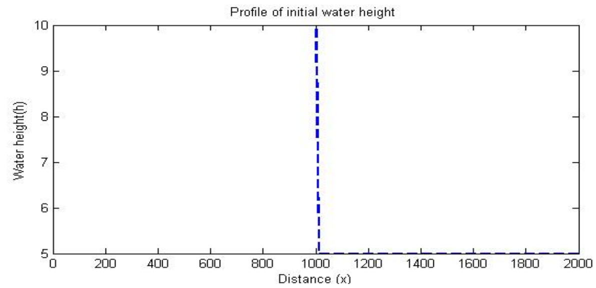

Figure 1: Initial Water height for dam break problem using height ratio 0.5.

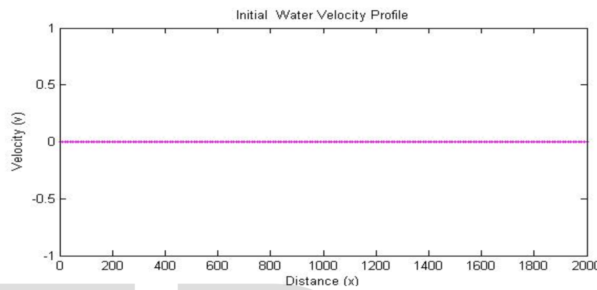

Figure 2: Initial water velocity for dam break problem using height ratio 0.5.

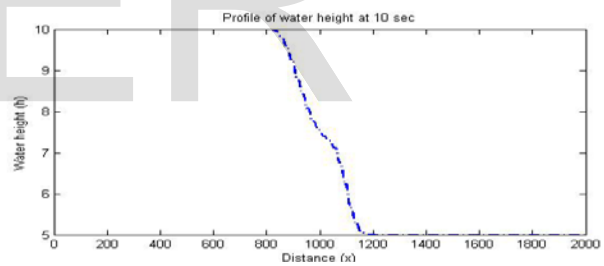

Figure 3: Water height for dam break at 10 sec using height ratio 0.5

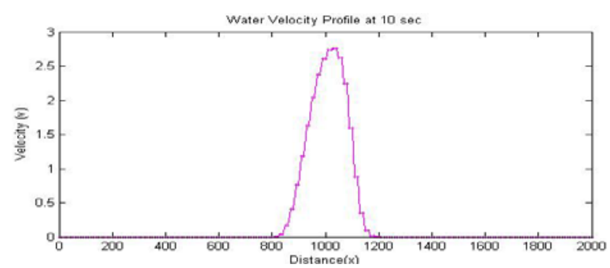

Figure 4: water velocity at 10 sec using height ratio 0.5

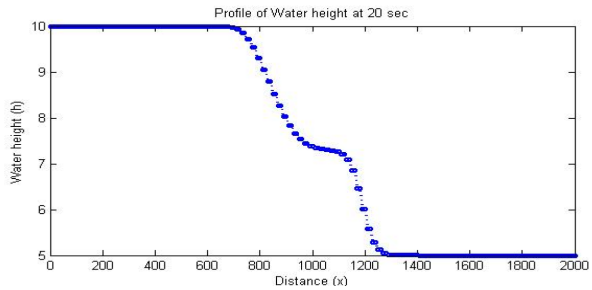

Figure 5: Water height at 20 sec using height ratio 0.5

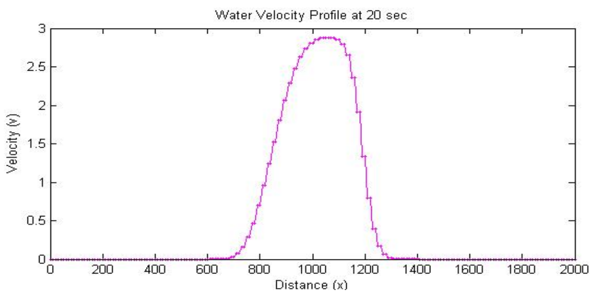

Figure 6: water velocity for dam break at 20 sec using height ratio 0.5

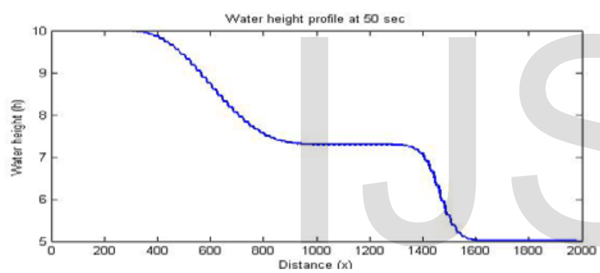

Figure 7: Water height at 50 sec using height ratio 0.5

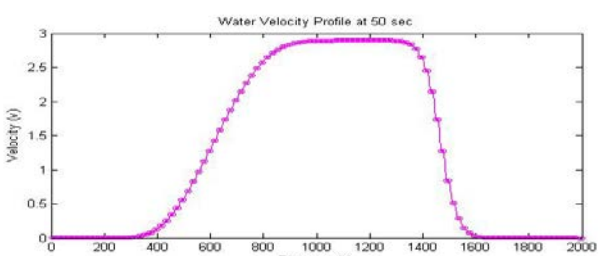

Figure 8: water velocity (right) for dam break at 50 sec using height ratio 0.5

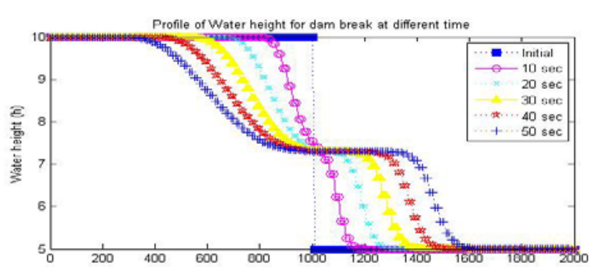

Figure 9: Water height for dam break at different time using height ratio 0.5.

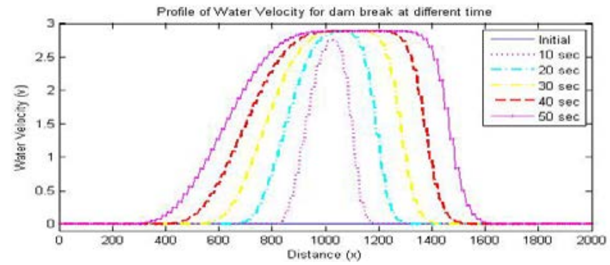

Figure 10: water velocity for dam break at different time using height ratio 0.5.

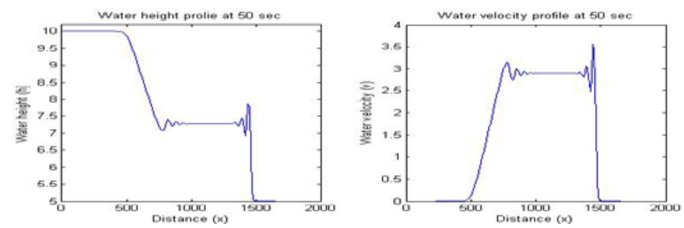

Figure 11: Water height (left) and water velocity (right) for dam break on wet bed at 50 sec using height ratio 0.5 by Lax-Wendroff method.

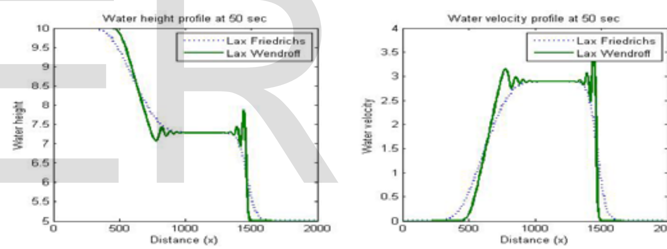

Figure 12: Comparative results for water height and water velocity in dam break at 50 sec.

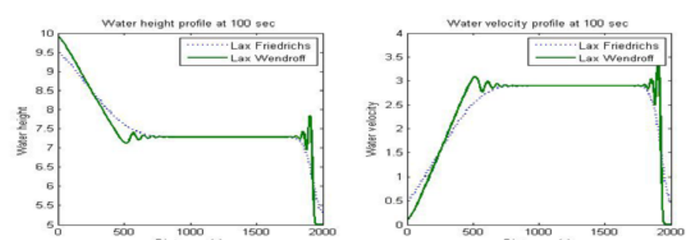

Figure 13: Comparative results for water height and water velocity in dam break at 100 sec using height ratio 0.5.

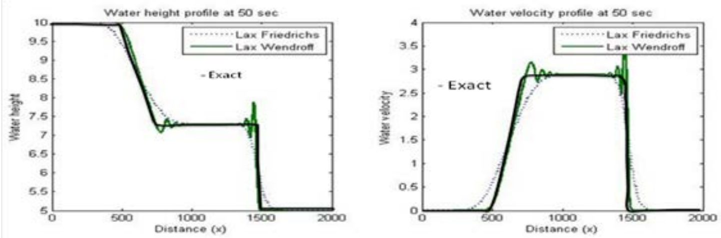

Figure 14: Lax Friedrichs method, Lax Wendroff method and analytic solution of the dam break test problem for 50 sec using height ratio 0.5.

Figure 3-10 shows the water height and velocity profiles at different times after the dam break flow problem by Lax Friedrichs method using a space discretization $\Delta x = 10m$. We have seen in figure 3, 5, 7 and 9 that water height of channel increases gradually towards the downstream direction and decreases gradually towards the upstream direction. We have also seen in figure 4, 6, 8, and 10 that velocity of channel increases gradually towards in both directions after the dam break. At time $t = 100s$, we observed in figure 13 that water height drops in the upstream boundary and water height rises in the downstream boundary for Lax Friedrichs and Lax Wendroff methods. We also observed in fig.13 that water velocity rises in both boundaries for Lax Friedrichs and Lax Wendroff methods. In fig.14, the comparison of numerical and analytical solution corresponding to water height as well as water velocity profile shows good agreement. The analytical reference solutions for these test problems are due to [3]. Therefore the above realistic phenomenons are well described by our implementation.

6 Conclusion

The study has presented the numerical solution of shallow water equations with dam break flow problem. We have studied Lax friedriches method and Lax Wendroff method for the numerical solution of shallow water equations. We have implemented the numerical scheme to simulate water depth and water velocity at different times and different points through shallow water dam break flow. The analysis of dam break flow is important to capture spatial and temporal evolution of flood event and safety analysis.

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